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Generalized Borel Law and Quantum Probabilities

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A version of the Borel law of large numbers is found which remains valid for nonclassical (Mackey-type) probability theories. Its relation to a frequency interpretation of quantum probabilities is discussed.

1. INTRODUCTION

Without entering into discussion about interpretations of probability, one could risk the claim that on the experimental level probabilities are always approximated by relative frequencies. So any form of probability theory becomes applicable to science only if it is able to express probabilities as limits of relative frequencies. The classical (Kolmogorov) probability theory meets this requirement, as it contains a collection of laws of large numbers (like the Borel law). There was nothing similar for the case of generalized (Mackey-type) probability theories based on structures weaker than Boolean algebras, except the standard quantum mechanics with its Borel–Finkelstein law. We fill this gap by demonstrating that some version of the classical Borel law holds in any Mackey-type probability theory, including the standard quantum mechanics, where it is equivalent to the Borel–Finkelstein law.

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2. TWO VERSIONS OF THE BOREL LAW

Let $[\Omega, \mathfrak{B}(\Omega), \mu]$ be a probability space, and $f_1, f_2, \ldots$ a sequence of identically distributed and mutually independent random variables on it. The well-known theorem of Kolmogorov (Gnedenko, 1976) states that

$$\mu\left\{\frac{1}{n}\sum_{k=1}^{n} f_k \to \bar{f}_1\right\} = 1$$

i.e., that the sequence $(1/n)\Sigma_{k=1}^{n} f_k$ converges to the mean value $\bar{f}_1$ almost certainly (μ−a.c.), if $\bar{f}_1$ does exist.

Let us fix a random variable f on $[\Omega, \mathfrak{B}(\Omega), \mu]$ and a Borel subset X of the real line $\mathbb{R}$. The relative frequency interpretation for $\mu(f^{-1}(X))$ is provided by the Borel theorem as follows. Let $\chi_{f^{-1}(X)}$ be the characteristic function of the random event $f^{-1}(X) \in \mathfrak{B}(\Omega)$. $\chi_{f^{-1}(X)}$ is a random variable on $[\Omega, \mathfrak{B}(\Omega), \mu]$, and its mean value $\bar{\chi}_{f^{-1}(X)}$ equals $\mu(f^{-1}(X))$. We take now the product probability space $[\hat{\Omega}, \mathfrak{B}(\hat{\Omega}), \hat{\mu}]$ (Halmos, 1950) with $\hat{\Omega} = \Omega^{\mathbb{N}}$, where $\mathbb{N}$ is the set of natural numbers, $\mathfrak{B}(\hat{\Omega})$ is σ-generated in $2^{\hat{\Omega}}$ in the standard way, and $\hat{\mu}$ is the product measure. The infinite sequences $(\omega_1, \omega_2, \ldots)$ which constitute $\hat{\Omega}$ represent long runs. The function χ_k: $\hat{\Omega} \to \mathbb{R}$, defined by $\chi_k(\omega_1, \omega_2, \ldots, \omega_k, \ldots) = 1$ if $\omega_k \in f^{-1}(X)$ and 0 otherwise, are random variables on the product probability space, and the sequence $\chi_1, \chi_2, \ldots$ fulfills the conditions of Kolmogorov's theorem. Thus the sequence $(1/n)\Sigma_{k=1}^{n}\chi_k$ of relative frequency functions converges to $\mu(f^{-1}(X))$ almost certainly as $n \to \infty$. This is the Borel law.

It is well known that any random variable, say f, on $[\Omega, \mathfrak{B}(\Omega), \mu]$ defines a σ-homomorphism f^{-1}: $\mathfrak{B}(\mathbb{R}) \to \mathfrak{B}(\Omega)$ of the Boolean σ-algebra $\mathfrak{B}(\mathbb{R})$ of Borel subsets of the real line into the Boolean σ-algebra $\mathfrak{B}(\Omega)$ of random events. Moreover almost all such σ-homomorphisms are generated by random variables. Thus in classical probability theory one can consider reverse random variables [the σ-homomorphisms of $\mathfrak{B}(\mathbb{R})$ into $\mathfrak{B}(\Omega)$] instead of random variables (real measurable functions on Ω).

For arbitrary fixed random variable f on $[\Omega, \mathfrak{B}(\Omega), \mu]$ and any $X \in \mathfrak{B}(\mathbb{R})$ we can formulate the Borel theorem in a slightly different manner. We define $\mu_{f^{-1}}$: $\mathfrak{B}(\mathbb{R}) \to [0,1]$ (the unit interval) by $\mu_{f^{-1}}(Y) = \mu(f^{-1}(Y))$ for any $Y \in \mathfrak{B}(\mathbb{R})$. $\mu_{f^{-1}}$ is a probability measure on $[\mathbb{R}, \mathfrak{B}(\mathbb{R})]$. Let χ_X be the characteristic function of X. χ_X is a random variable on $[\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu_{f^{-1}}]$, and its mean value $\bar{\chi}_X$ equals $\mu_{f^{-1}}(X) = \mu(f^{-1}(X))$. We construct the product space $[\hat{\mathbb{R}}, \mathfrak{B}(\hat{\mathbb{R}}), \hat{\mu}_{f^{-1}}]$ in the same way as previously. Elements of $\hat{\mathbb{R}}$, i.e., sequences $(x_1, x_2, \ldots)$ of real numbers represent now results of long runs.

Functions χ'_k, defined by $\chi'_k(x_1, x_2, \ldots, x_k, \ldots) = 1$ if $x_k \in X$, and 0 otherwise, form a sequence of random variables on $[\hat{\mathbb{R}}, \mathfrak{B}(\hat{\mathbb{R}}), \hat{\mu}_{f^{-1}}]$ which fulfill the assumptions of Kolmogorov's theorem. Hence the sequence $(1/n)\Sigma_{k=1}^{n} \chi'_k$ of relative frequency functions converges to $\bar{\chi}'_1 = \bar{\chi}_X = \mu(f^{-1}(X))$ almost certainly as $n \to \infty$. This version of the Borel theorem is much more capable of generalization than the previous one.

The equivalence of the two versions of the Borel law should be understood as the commutativity of the following diagram:

$$\begin{tikzcd}
& \mathfrak{B}(\mathbb{R}) \arrow[dl, "\chi_k^{-1}"'] \arrow[dr, "\chi_k'^{-1}"] & \\
\mathfrak{B}(\hat{\Omega}) \arrow[dr, "\hat{\mu}"'] & & \mathfrak{B}(\hat{\mathbb{R}}) \arrow[ll, "h"'] \arrow[dl, "\hat{\mu}_{f^{-1}}"] \\
& {[0,1]} &
\end{tikzcd}$$

with h being a σ-homomorphism, for any $k = 1, 2, \ldots$. The mapping h, defined by $h(\chi_k'^{-1}(1)) = \chi_k^{-1}(1)$, $h(\chi_k'^{-1}(0)) = \chi_k^{-1}(0)$ for all k, can be extended to a σ-homomorphism of $\mathfrak{B}(\hat{\mathbb{R}})$ into $\mathfrak{B}(\hat{\Omega})$.

3. BOREL LAW IN GENERALIZED PROBABILITY THEORIES

Observe that our second version of the Borel theorem does not make use of specific structure of $[\Omega, \mathfrak{B}(\Omega), \mu]$. We could dispense with the set Ω of elementary events, and consider another (Onicescu, 1973) variant of the classical probability theory based on an abstract Boolean σ-algebra $\mathfrak{B}$ together with a normal measure μ on $\mathfrak{B}$. Moreover, we could go further and relax also the Boolean structure of the set of random events to obtain a far-reaching generalization of the classical probability theory. Indeed, it has been indicated by Mackey (Mackey, 1963) [see also papers on "quantum logic," e.g., Greechie and Gudder (1973), Jauch (1974), Beltrametti and Cassinelli (1976)] that an orthomodular σ orthoadditive ortho-p.o.set (σ-ortho-p.o.set for short) can do the job of $\mathfrak{B}(\Omega)$. This is one way to obtain a nonclassical ("noncommutative") probability theory.

Thus let $\mathfrak{L}$ be a σ-ortho-p.o.set with the greatest element e, and $a \to -a$, the orthocomplementation. Probability measure on $\mathfrak{L}$ is a σ-additive function μ: $\mathfrak{L} \to [0,1]$ such that $\mu(e) = 1$. A random variable A (it corresponds to the "classical" reverse random variable) is an $\mathfrak{L}$-valued measure, i.e., a mapping of $\mathfrak{B}(\mathbb{R})$ into $\mathfrak{L}$ such that $A(\mathbb{R})=e$, if $X_1 \cap X_2 = \varnothing$ then $A(X_1) \leqslant -A(X_2)$ for any $X_1, X_2 \in \mathbb{R}$, and $A(X_1 \cup X_2 \cdots) = A(X_1) + A(X_2) + \cdots$ for any $X_1, X_2, \ldots \in \mathfrak{B}(\mathbb{R})$ with $X_i \cap X_j = \varnothing$, $i \neq j$. A pair $[\mathfrak{L}, \mu]$ plays the role of (generalized) probability space and, as previously, any random variable A generates a measure μ_A on $[\mathbb{R}, \mathfrak{B}(\mathbb{R})]$. The probability space $[\mathbb{R}, \mathfrak{B}(\mathbb{R}), \mu_A]$ obtained in this manner is classical, and we can repeat the construction leading to the Borel theorem without any changes. Thus we obtain $\mu(A(X))$ as an a.c. limit of a sequence of relative frequency functions. This is a general ("nonclassical") version of the Borel theorem.

The probability theory based on a σ-ortho-p.o.set is so general that it covers also the case of standard quantum theory. For this case $\mathfrak{L}$ is identified as the complete ortholattice $\mathfrak{L}(\mathcal{H})$ of all closed subspaces of a Hilbert space $\mathcal{H}$, probability measures on $\mathfrak{L}(\mathcal{H})$ are represented by statistical operators (the Gleason theorem), and random variables ($\mathfrak{L}(\mathcal{H})$-valued measures on $[\mathbb{R}, \mathfrak{B}(\mathbb{R})]$) are just spectral decompositions of quantum "observables." Our remarks above apply without any changes to this case. [A similar idea has been reported by Ochs (1980); see also Lahti and Talja (1980).]

In the traditional language of quantum mechanics the general version of the Borel theorem goes as follows: A is a fixed "observable" (a self-adjoint operator densely defined on a separable Hilbert space $\mathcal{H}$) representing a measurable physical quantity, and the real line $\mathbb{R}$ contains the set of all possible values (the set of all possible results of single measurements) of the quantity (the spectrum of A). The (generalized) probability measure μ corresponds to a physical state (a preparation procedure), whereas μ_A is the probability measure on $\mathbb{R}$ resulting from a long sequence of single measurements of A. The product space $\hat{\mathbb{R}} = \mathbb{R}^{\mathbb{N}}$ contains all results of such sequences of single measurements. Accordingly to the (generalized) Borel theorem the relative frequency functions $(1/n)\Sigma_{k=1}^{n} \chi_k'$ converge almost certainly to $\bar{\chi}_X = \mathrm{Tr}(\rho_\mu P_{A,X})$, where ρ_μ is the statistical operator corresponding to μ, and $P_{A,X}$ is the projection operator related to the Borel set $X \subset \mathbb{R}$ via the spectral decomposition of A. This is a formal demonstration of the correctness of a fact that is obvious to any experimental physicist, namely, that quantum probabilities can be approximated by relative frequencies. On the other hand this does not mean that probabilities are relative frequencies [cf. the clear analysis in van Fraassen, (1977)], and does not exclude the propensity interpretation having some advantages for quantum probabilities (cf. Popper, 1967; Lahti and Talja, 1980).

The first version of the Borel law cannot be formulated for the general probability theory based on a σ-ortho-p.o.set $\mathfrak{L}$ as there is no way to construct a product of $\mathfrak{L}$'s generalizing $\mathfrak{B}(\hat{\Omega})$. In the special case of standard quantum mechanics, however, the σ-ortho-p.o.set $\mathfrak{L}(\hat{\mathcal{H}})$ of closed subspaces of the von Neumann tensor product $\hat{\mathcal{H}}$ of countable infinite number of $\mathcal{H}$'s (von Neumann, 1938) is a legitimate counterpart of $\mathfrak{B}(\hat{\Omega})$. The product measure $\hat{\mu}$ is here represented by the infinite tensor product $\rho_{\hat{\mu}}=\rho_\mu\otimes\rho_\mu\otimes\cdots$ of statistical operators ρ_μ corresponding to a probability measure μ on $\mathfrak{L}(\mathcal{H})$. The striking analogy of $\mathfrak{L}(\hat{\mathcal{H}})$ to $\mathfrak{B}(\hat{\Omega})$ suggests a formulation of the first version of the Borel law in terms of $\mathfrak{L}(\hat{\mathcal{H}})$ and $\rho_{\hat{\mu}}$. This has been done by Finkelstein (Finkelstein, 1965) and rediscovered by Hartle (Hartle, 1968) (see also Davidon, 1976; d'Espagnat, 1976; Ochs, 1977).

The remarkable Finkelstein translation of the Borel law into the Hilbert space language could be formulated as follows: Let A be a fixed quantum observable on a Hilbertian "probability space" $[\mathcal{H}, \mathfrak{L}(\mathcal{H}), \mu]$, and let $X\in\mathfrak{B}(\mathbb{R})$. The projection operator $P_{A,X}$ related to X by the spectral measure of A corresponds to $f^{-1}(X)$ in classical Borel law. We take the "product probability space" $[\hat{\mathcal{H}}, \mathfrak{L}(\hat{\mathcal{H}}), \hat{\mu}]$ and define projection operators P_k on $\hat{\mathcal{H}}$ by $P_k=E\otimes E\otimes\cdots\otimes P_{A,X}\otimes E\otimes\cdots$, where E is the identity operator on $\mathcal{H}$ and $P_{A,X}$ occupies the kth position.

Now it can be proved that the sequence $(1/n)\sum_{k=1}^n P_k$ converges $\hat{\mu}-$a.c. to $[\mathrm{Tr}(\rho_\mu P_{A,X})]\hat{I}$ ($\hat{I}$ is the identity operator on $\hat{\mathcal{H}}$ in the usual sense of convergences of spectral measures (Ochs, 1977). This is the Borel–Finkelstein law.

Our variant of Borel law for quantum mechanics is connected to the above result in the same way as the second version of the classical Borel law is related to the first one. The diagram

$$\begin{tikzcd}
& \mathfrak{B}(\mathbb{R}) \arrow[dl, "P_k"'] \arrow[dr, "\chi_k'^{-1}"] & \\
\mathfrak{L}(\hat{\mathcal{H}}) \arrow[dr, "\hat{\mu}"'] & & \mathfrak{B}(\hat{\mathbb{R}}) \arrow[ll, "h"'] \arrow[dl, "\hat{\mu}_A"] \\
& {[0,1]} &
\end{tikzcd}$$

is commutative for all k, with a σ-homomorphism h generated via the Varadarajan theorem (Varadarajan, 1962, Theorem 3.4) by the mapping $h(\chi_k'^{-1}(1))=P_k$, $h(\chi_k'^{-1}(0))=P_k^\perp$ (the ortho-complement of P_k).

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